Completions of mapping class groups and the cycle $C - C^-$

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1 Introduction

The classical Malcev (or unipotent completion) of an abstract group π is a prounipotent group $\mathcal{P}$ defined over $\mathbb{Q}$ together with a homomorphism $\phi : \pi \to \mathcal{P}$. It is characterized by the property that if $\psi : \pi \to \mathcal{U}$ is a homomorphism of π into a prounipotent group, then there is a unique homomorphism $\Psi : \mathcal{P} \to \mathcal{U}$ of prounipotent groups such that $\psi = \Psi \phi$.

$$
\begin{array}{ccc}
\pi & \xrightarrow{\phi} & \mathcal{P} \\
& \psi \searrow & \downarrow \Psi \\
& & \mathcal{U}
\end{array}
$$

When $H_1(\pi; \mathbb{Q}) = 0$, the unipotent completion is trivial, and it is therefore a useless tool for studying mapping class groups. Deligne has suggested a notion of *relative Malcev completion*: Suppose Γ is an abstract group and that $\rho : \Gamma \to S$ is a homomorphism of Γ into a semisimple linear algebraic group defined over $\mathbb{Q}$. Suppose that ρ has Zariski dense image. The *completion of Γ relative to ρ* is a proalgebraic group $\mathcal{G}$ over $\mathbb{Q}$, which is an extension of S by a prounipotent group $\mathcal{U}$, and a homomorphism $\tilde{\rho} : \Gamma \to \mathcal{G}$ which lifts ρ. When S is the trivial group, it reduces to the unipotent completion. The relative completion is characterized by a universal mapping property which generalizes the one in the unipotent case (see (2.3)).

Denote the mapping class group associated to a surface of genus g with r boundary components and n ordered marked distinct points by $\Gamma_{g,r}^n$. The mapping class group $\Gamma_{g,r}^n$ has a natural representation $\rho : \Gamma_{g,r}^n \to Sp_g(\mathbb{Z})$ obtained from the action of $\Gamma_{g,r}^n$ on the first homology group of the underlying compact Riemann surface. Its kernel is, by definition, the Torelli group $T_{g,r}^n$. One can therefore form the completion of $\Gamma_{g,r}^n$ relative to ρ. It is a proalgebraic group $\mathcal{G}_{g,r}^n$ which is an extension

$$
1 \to \mathcal{U}_{g,r}^n \to \mathcal{G}_{g,r}^n \to Sp_g \to 1
$$

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of Sp_g by a prounipotent group. The homomorphism

$$\tilde{\rho} : \Gamma_{g,r}^n \to \mathcal{G}_{g,r}^n$$

induces a homomorphism $T_{g,r}^n \to \mathcal{U}_{g,r}^n$. This, in turn, induces a homomorphism $\mathcal{T}_{g,r}^n \to \mathcal{U}_{g,r}^n$ from the unipotent completion of the Torelli group into $\mathcal{U}_{g,r}^n$. Our main result is:

Theorem. *When $g \geq 3$, the natural homomorphism $\mathcal{T}_{g,r}^n \to \mathcal{U}_{g,r}^n$ is surjective with nontrivial kernel which is contained in the center of $\mathcal{T}_{g,r}^n$ and which is isomorphic to $\mathbb{Q}$ whenever $g \geq 8$.*

We prove this by relating the central extension above to the line bundle over the moduli space of genus three curves associated to the archimedean height of the algebraic cycle $C - C^-$ in the jacobian of a curve C of genus 3.[2] This theorem is related to, and complements, the work of Morita [21]. The constant 8 in the theorem can surely be improved, possibly to 3.[3]

One reason for introducing relative completions of fundamental groups of varieties, and of mapping class groups in particular, is that their coordinate rings are, under suitable conditions, direct limits of variations of mixed Hodge structure over the variety. This result and some of its applications to the action of the mapping class group of a surface S on the lower central series of $\pi_1(S, *)$ will be presented elsewhere.

Part of a general theory of relative completions is worked out in Section 4. Many of the results of that section were worked out independently and contemporaneously by Eduard Looijenga. I would like to thank him for his correspondence. I would also like to thank P. Deligne for his correspondence, and the Mathematics Department of the University of Utrecht for its hospitality and support during a visit in May, 1992 when this paper was written.

Conventions: The group $Sp_g(R)$ will denote the group of automorphisms of a free R module of dimension $2g$ which preserve a unimodular skew symmetric bilinear form. In short, elements of $Sp_g(R)$ are $2g \times 2g$ matrices.

2 Relative completion

Fix a field F of characteristic zero. Suppose that π is a abstract group and that $\rho : \pi \to S$ is a representation of π into a linear algebraic group S defined over F. Assume that the image of ρ is Zariski dense. In this section we define the *completion of π relative to ρ*. When S is the trivial group, this reduces

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[2] When the genus g of C is ≥ 3, one can relate this central extension to the height pairing between the cycles $C^{(a)} - C^{(a)-}$ and $C^{(b)} - C^{(b)-}$ in $\mathrm{Jac}\, C$, where $a + b = g - 1$ and $C^{(r)}$ denotes the rth symmetric power of C. We chose not to do this in order to keep the Hodge theory straightforward.

[3] The optimal constant is the smallest integer d such that $H^2(Sp_g(\mathbb{Z}), A)$ vanishes for all rational representations A of $Sp_g(\mathbb{Q})$ whenever $g \geq d$.

to the Malcev completion (a.k.a. unipotent completion) which is defined, for example, in [24], [3] and [25]. The idea of relative completion is due to Deligne [4] and is a refinement of the idea of the "algebraic hull" of a group introduced by Hochschild and Mostow [14, p. 1140].

To construct the relative completion of π with respect to ρ, consider all commutative diagrams of the form

$$
\begin{array}{ccccccccc}
1 & \to & U & \to & E & \to & S & \to & 1 \\
 & & & & \tilde{\rho} \uparrow & \nearrow \rho & & & \\
 & & & & \pi & & & &
\end{array}
$$

where E is a linear algebraic group over F, U a unipotent subgroup of E, and where $\tilde{\rho}$ is a lift of ρ to E whose image is Zariski dense. All morphisms in the top row are algebraic group homomorphisms. One can define morphisms of such diagrams in the obvious way.

Proposition 2.1 *The set of such diagrams forms an inverse system.*

Proof. If

$$
\begin{array}{ccccccccc}
1 & \to & U_\alpha & \to & E_\alpha & \to & S & \to & 1
\end{array}
$$

$$
\begin{array}{ccccccccc}
1 & \to & U_\beta & \to & E_\beta & \to & S & \to & 1
\end{array}
$$

are two extensions of S by a unipotent algebraic group, then one can form the fibered product

$$
\begin{array}{ccc}
E & \to & E_\alpha \\
\downarrow & & \downarrow \\
E_\beta & \to & S.
\end{array}
$$

The natural homomorphism $E \to S$ is surjective with kernel the unipotent group $U_\alpha \times U_\beta$.

Now suppose that $\rho_\alpha : \pi \to E_\alpha$ and $\rho_\beta : \pi \to E_\beta$ are lifts of $\rho : \pi \to S$ to E_α and E_β, respectively, both with Zariski dense image. They induce a homomorphism $\rho_{\alpha\beta} : \pi \to E$ which lifts both ρ_α and ρ_β. Denote the Zariski closure of the image of $\rho_{\alpha\beta}$ in E by $E_{\alpha\beta}$. Then $E_{\alpha\beta}$ is a linear algebraic group and the kernel of the natural homomorphism $E_{\alpha\beta} \to S$ is unipotent as it is a subgroup of $U_\alpha \times U_\beta$. The natural map $E_{\alpha\beta} \to S$ is surjective as the image of ρ is Zariski dense in S. $\quad\square$

Definition 2.2 The *completion* $\mathcal{P}_F$ *of* π *(over F) relative to* $\rho : \pi \to S$ is defined to be the proalgebraic group

$$
\mathcal{P}_F = \varprojlim E,
$$

where the inverse limit is taken over all commutative diagrams

$$
\begin{array}{ccccccccc}
1 & \to & U & \to & E & \to & S & \to & 1 \\
 & & & & \tilde{\rho} \uparrow & \rho \nearrow & & & \\
 & & & & \pi & & & &
\end{array}
$$

whose top row is an extension of S by a unipotent group in the category of linear algebraic groups over F, and where $\tilde{\rho}$ has dense image. The homomorphisms $\tilde{\rho}: \pi \to E$ induce a canonical homomorphism $\pi \to \mathcal{P}_F$.

Often we will simply say that $\pi \to \mathcal{P}_F$ is the S-*completion* of π. The coordinate ring of $\mathcal{P}_F$ is the direct limit of the coordinate rings of the groups E. It will be denoted $\mathcal{O}(\mathcal{P}_F)$. It is a commutative Hopf algebra with antipode.

There is a natural surjection $\mathcal{P}_F \to S$ whose kernel is a prounipotent group. When S is the trivial group, we obtain the classical Malcev completion. The S-Malcev completion is characterized by the following easily verified universal mapping property.

Proposition 2.3 *Suppose that $\mathcal{E}$ is a linear proalgebraic group defined over F, and that $\mathcal{E} \to S$ is a homomorphism of proalgebraic groups with prounipotent kernel. If $\varphi : \pi \to \mathcal{E}$ is a group homomorphism, then there is a unique homomorphism $\tau : \mathcal{P}_F \to \mathcal{E}$ of pro-algebraic groups over F such that the diagram*

$$
\begin{array}{ccc}
& \mathcal{P}_F & \\
\hat{\rho} \nearrow & \searrow & \\
\pi & \downarrow \tau & S \\
\varphi \searrow & \nearrow & \\
& \mathcal{E} &
\end{array}
$$

commutes. □

Suppose that G is a (pro)algebraic group over the field F. Suppose that k is a field extension of F. We shall denote G, viewed as an algebraic group over k by extension of scalars, by $G(k)$. The following assertion follows directly from the universal mapping property.

Corollary 2.4 *If k is a field extension of the field F, then there is a natural homomorphism $\mathcal{P}_k \to \mathcal{P}_F(k)$.* □

We will show in the next two sections that, with some extra hypotheses, this homomorphism is always an isomorphism.

3 A construction of the Malcev completion

There is an explicit algebraic construction of the Malcev completion which is due to Quillen [24]. We will use it to show that the Malcev completion of a group over F is isomorphic to the F points of its Malcev completion over $\mathbb{Q}$.

Denote the group algebra of a group π over a commutative ring R by $R\pi$. The *augmentation* is the homomorphism $\epsilon : R\pi \to R$ defined by taking each $\gamma \in \pi$ to 1. The kernel of the augmentation is called the *augmentation ideal* and will be denoted by J_R, or simply J when there is no chance of confusion.

With the coproduct $\Delta : R\pi \to R\pi \otimes R\pi$ defined by $\Delta(\gamma) = \gamma \otimes \gamma$, for all $\gamma \in \pi$, $R\pi$ has the structure of a cocommutative Hopf algebra.

The powers of the augmentation ideal define a topology on $R\pi$ that is called the *J-adic topology*. The *J-adic completion* of the group ring is the R module

$$R\pi\widehat{\ } := \varprojlim R\pi/J^l.$$

The completion of J will be denoted by $\widehat{J}$. Since the coproduct is continuous, it induces a coproduct

$$\Delta : R\pi\widehat{\ } \to R\pi\widehat{\otimes}R\pi,$$

where $\widehat{\otimes}$ denotes the completed tensor product. This gives $R\pi\widehat{\ }$ the structure of a complete Hopf algebra.

The proof of the following proposition is straightforward.

Proposition 3.1 *If π is a group and R a ring, then the function $\pi \to J_R/J_R^2$ defined by taking $\gamma \in \pi$ to the coset of $\gamma - 1$, induces an R-module isomorphism*

$$H_1(\pi, R) \approx J_R/J_R^2. \quad \square$$

Now let R be a field F of characteristic zero. The logarithm and exponential maps are mutually inverse homeomorphisms

$$\log : 1 + \widehat{J}_F \to \widehat{J}_F \text{ and } \exp : \widehat{J}_F \to 1 + \widehat{J}_F.$$

The set of *primitive elements* $\mathfrak{p}$ of $F\pi\widehat{\ }$ is defined by

$$\mathfrak{p} = \{X \in \widehat{J}_F : \Delta(X) = X \otimes 1 + 1 \otimes X\}.$$

With the bracket $[X, Y] = XY - YX$, it has the structure of a Lie algebra. The topology of $F\pi\widehat{\ }$ induces a topology on $\mathfrak{p}$, giving it the structure of a complete topological Lie algebra.

The set of *group-like* elements $\mathcal{P}$ of $F\pi\widehat{\ }$ is defined by

$$\mathcal{P} = \{X \in F\pi\widehat{\ } : \Delta(X) = X \otimes X \text{ and } \epsilon(X) = 1\}.$$

It is a subgroup of the group of units of $F\pi\widehat{\ }$. The logarithm and exponential maps restrict to mutually inverse homeomorphisms

$$\log : \mathcal{P} \to \mathfrak{p} \text{ and } \exp : \mathfrak{p} \to \mathcal{P}.$$

The filtration of $F\pi\widehat{\ }$ by the powers of J induces filtrations of $\mathcal{P}$ and $\mathfrak{p}$: Set

$$\mathfrak{p}^l = \mathfrak{p} \cap \widehat{J}^l \text{ and } \mathcal{P}^l = \mathcal{P} \cap (1 + \widehat{J}^l).$$

These satisfy

$$\mathfrak{p} = \mathfrak{p}^1 \supseteq \mathfrak{p}^2 \supseteq \mathfrak{p}^3 \cdots$$

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and
$$\mathcal{P} = \mathcal{P}^1 \supseteq \mathcal{P}^2 \supseteq \mathcal{P}^3 \cdots.$$

These filtrations define topologies on $\mathfrak{p}$ and $\mathcal{P}$. Both are separated and complete. For each l set
$$\mathfrak{p}_l = \mathfrak{p}/\mathfrak{p}^{l+1} \text{ and } \mathcal{P}_l = \mathcal{P}/\mathcal{P}^{l+1}.$$

It follows easily from (3.1) that if $H_1(\pi, \mathbb{Q})$ is finite dimensional (e.g. π is finitely generated), then each $\mathcal{P}_l$ is a linear algebraic group.

Since the logarithm and exponential maps induce isomorphisms between $1 + \widehat{J}/\widehat{J}^{l+1}$ and $\widehat{J}/\widehat{J}^{l+1}$, and since

$$\mathfrak{p}_l \subseteq \widehat{J}/\widehat{J}^{l+1} \text{ and } \mathcal{P}_l \subseteq 1 + \widehat{J}/\widehat{J}^{l+1},$$

it follows that the logarithm and exponential maps induce polynomial bijections between $\mathfrak{p}_l$ and $\mathcal{P}_l$. Consequently, when $H_1(\pi, \mathbb{Q})$ is finite dimensional, each of the groups $\mathcal{P}_l$ is a unipotent algebraic group over F with Lie algebra $\mathfrak{p}_l$. It follows that if $H_1(\pi, F)$ is finite dimensional, then $\mathcal{P}$ is a prounipotent group over F.

The composition of the canonical inclusion of π into $F\pi$ followed by the completion map $F\pi \to \widehat{F\pi}$ yields a canonical map $\pi \to \widehat{F\pi}$. Since the image of this map is contained in $\mathcal{P}$, there is a canonical homomorphism $\pi \to \mathcal{P}$. The composition of the natural homomorphism $\pi \to \mathcal{P}$ with the quotient map $\mathcal{P} \to \mathcal{P}_l$ yields a canonical homomorphism $\pi \to \mathcal{P}_l$.

Proposition 3.2 *If $H_1(\pi, F)$ is finite dimensional, then each of the homomorphisms $\pi \to \mathcal{P}_l$ has Zariski dense image.*

Proof. Denote the Zariski closure of the image of π in $\mathcal{P}_l$ by $\mathcal{Z}_l$. Each $\mathcal{Z}_l$ is an algebraic subgroup of $\mathcal{P}_l$. Since the composite

$$H_1(\pi; F) \to H_1(\mathcal{Z}_l) \to H_1(\mathcal{P}_l)$$

is an isomorphism, it follows that the second map is surjective. Since $\mathcal{P}_l$ is unipotent, this implies that the inclusion $\mathcal{Z}_l \hookrightarrow \mathcal{P}_l$ is surjective. $\square$

Theorem 3.3 *If $H_1(\pi, F)$ is finite dimensional, then the natural map $\pi \to \mathcal{P}$ is the Malcev completion of π over F.*

Proof. By the universal mapping property (2.3), there is a canonical homomorphism from the Malcev completion $\mathcal{U}_F$ of π to $\mathcal{P}$. It follows from (3.2) that this homomorphism is surjective.

We now establish injectivity. Suppose that U is a unipotent group defined over F. This means that U can be represented as a subgroup of the group of upper triangular unipotent matrices in $GL_n(F)$ for some n. A representation $\rho : \pi \to U$ induces a ring homomorphism

$$\tilde{\rho} : F\pi \to gl_n(F).$$

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Since the representation is unipotent, it follows that $\tilde{\rho}(J)$ is contained in the set of nilpotent upper triangular matrices. This implies that the kernel of $\tilde{\rho}$ contains J^n. Consequently, ρ induces a homomorphism

$$\bar{\rho} : F\pi/J^n \to gl_n(F).$$

Since the image of J is contained in the nilpotent upper triangular matrices, the image of the subgroup $\mathcal{P}_{n-1}$ of $1 + J/J^n$ is contained in the group of unipotent upper triangular matrices. Because the image of π is Zariski dense in $\mathcal{P}_{n-1}$, it follows that the image of $\mathcal{P}_{n-1}$ is contained in U. That is, there is a homomorphism $\mathcal{P}_{n-1} \to U$ of linear algebraic groups over F such that the diagram

$$\begin{array}{ccc} \pi & \to & \mathcal{P}_{n-1} \\ {\scriptstyle\rho}\searrow & & \downarrow \\ & & U \end{array}$$

commutes. It follows that $\mathcal{U}_F \to \mathcal{P}$ is injective. $\qquad\square$

Corollary 3.4 *If $\mathcal{P}_F$ is the Malcev completion of the group π over F, then the natural homomorphism*

$$\mathcal{P}_F \to \mathcal{P}_{\mathbb{Q}}(F)$$

is an isomorphism.

Proof. This follows as $\mathcal{P}_F$ is the set of group-like elements of $F\widehat{\pi}$, while $\mathcal{P}_{\mathbb{Q}}(F)$ is the set of group-like elements of $\mathbb{Q}\widehat{\pi} \otimes F \approx F\widehat{\pi}$. $\qquad\square$

4 Basic theory of relative completion

In this section we establish several basic properties of relative completion. We begin by recalling some basic facts about group extensions. Once again, F will denote a fixed field of characteristic zero. All algebraic groups will be linear.

Suppose that L is an abstract group and that A is an L module. The group of congruence classes of extensions

$$0 \to A \to G \to L \to 1,$$

where the natural action of L on A is the given action, is naturally isomorphic to $H^2(L; A)$. The identity is the semidirect product $L \ltimes A$ ([20, Theorem 4.1, p. 112]). If $H^1(L; A)$ vanishes, then any 2 splittings $s_0, s_1 : L \to L \ltimes A$ are conjugate via an element of A ([20, Prop. 2.1, p. 106]). That is, there exists $a \in A$ such that $s_1 = as_0a^{-1}$.

If S is a connected semisimple algebraic group over F, and if A is a rational representation of S, then every extension

$$0 \to A \to G \to S \to 1$$

in the category of algebraic groups over F splits. Moreover, any 2 splittings $s_0, s_1 : S \to G$ are conjugate by an element of A [15, p. 185].

These results extend to the case when the kernel is unipotent. For this we need the following construction.

(4.1) **Construction.** Suppose that

$$1 \to U \to G \to L \to 1 \qquad (1)$$

is an extension of an abstract group L by a group U. Suppose that Z is a central subgroup of U and that the extension

$$1 \to U/Z \to G/Z \to L \to 1 \qquad (2)$$

is split. Let $s : L \to G/Z$ be a splitting. Pulling back the extension

$$1 \to Z \to G \to G/Z \to 1$$

along s, one obtains an extension

$$1 \to Z \to E_s \to L \to 1. \qquad (3)$$

This determines, and is determined by, a class ζ_s in $H^2(L; Z)$ which depends only on s up to inner automorphisms by elements of G. The extension (3) splits if and only if $\zeta_s = 0$.

Proposition 4.2 *If $\zeta_s = 0$, then the extension (1) splits. Moreover, if any 2 splittings of (2) are conjugate, and if $H^1(L; Z) = 0$, then any 2 splittings of (1) are conjugate.*

Proof. If ζ_s vanishes, there is a splitting $\sigma : L \to E_s$. By composing σ with the inclusion of $E_s \hookrightarrow G$, one obtains a splitting of the extension (1).

Suppose now that $H^1(L; Z)$ vanishes, and that any 2 splittings of $L \to G/Z$ are conjugate. If $s_0, s_1 : L \to G$ are 2 splittings of (1), then their reductions $\bar{s}_0, \bar{s}_1 : L \to G/Z$ mod Z are conjugate. We may therefore assume that s_0 and s_1 agree mod Z. The images of s_0 and s_1 are then contained in the subgroup E_s of G determined by the section $\bar{s}_0 = \bar{s}_1 : L \to G/Z$. Since $H^1(L; Z) = 0$, there is an element z of Z which conjugates s_0 into s_1. $\square$

A similar argument, combined with the facts about extensions of algebraic groups at the beginning of this section, can be used to prove the following result.

Proposition 4.3 *Suppose that S is a connected semisimple algebraic group over F. If*

$$1 \to U \to G \to S \to 1$$

is an extension in the category of algebraic groups over F, and if U is unipotent, then the extension splits, and any two splittings $S \to G$ are conjugate. $\square$

By choosing compatible splittings and then taking inverse limits, we obtain the analogous result for proalgebraic groups.

Proposition 4.4 *Suppose that S is a connected semisimple algebraic group over F. If*

$$1 \to \mathcal{U} \to \mathcal{G} \to S \to 1$$

is an extension in the category of proalgebraic groups over F, and if $\mathcal{U}$ is prounipotent, then the extension splits, and any two splittings $S \to \mathcal{G}$ are conjugate. $\square$

Suppose that Γ is a abstract group, S an algebraic group over F, and that $\rho : \Gamma \to S$ a representation with Zariski dense image. Denote the image of ρ by L and its kernel by T. Thus we have an extension

$$1 \to T \to \Gamma \to L \to 1.$$

Let $\Gamma \to \mathcal{G}_F$ be the completion over F of Γ relative to ρ and $\mathcal{U}_F$ its prounipotent radical. There is a commutative diagram

$$
\begin{array}{ccccccccc}
1 & \to & T & \to & \Gamma & \to & L & \to 1 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
1 & \to & \mathcal{U}_F & \to & \mathcal{G}_F & \to & S & \to 1
\end{array}
$$

Denote the unipotent (i.e., the Malcev) completion of T over F by $\mathcal{T}_F$. The universal mapping property of $\mathcal{T}_F$ gives a homomorphism $\Phi : \mathcal{T}_F \to \mathcal{U}_F$ of prounipotent groups whose composition with the natural map $T \to \mathcal{T}_F$ is the canonical map $T \to \mathcal{U}_F$. Denote the kernel of Φ by $\mathcal{K}_F$.

Proposition 4.5 *Suppose that $H_1(T; F)$ is a finite dimensional. If the action of L on $H_1(T; F)$ extends to a rational action of S, then the kernel $\mathcal{K}_F$ of Φ is central in $\mathcal{T}_F$.*

Proof. First, Γ acts on the completion $FT\widehat{\ }$ of the group algebra of T by conjugation. This action preserves the filtration by the powers of $\widehat{J}$, so it acts on the associated graded algebra

$$Gr^{\bullet}_J FT\widehat{\ } = \bigoplus_{m=0}^{\infty} \widehat{J}^m / \widehat{J}^{m+1}.$$

If $H_1(T; F)$ is finite dimensional, each truncation FT/J^l of $FT\widehat{\ }$ is finite dimensional. This implies that each of the groups $\operatorname{Aut} FT/J^l$ is an algebraic group. Since $\operatorname{Aut} Gr^{\bullet}_J FT\widehat{\ }$ is generated by $J/J^2 = H_1(T; F)$, it follows that when $H_1(T; F)$ is finite dimensional, $\operatorname{Aut} FT\widehat{\ }$, the group of augmentation preserving algebra automorphisms of $FT\widehat{\ }$, is a proalgebraic group which is an extension of a subgroup of $\operatorname{Aut} H_1(T; F)$ by a prounipotent group.

If the action of Γ on $H_1(T; F)$ factors through a rational representation $S \to \operatorname{Aut} H_1(T; F)$ of S, then we can form a proalgebraic group extension

$$1 \to J^{-1}\operatorname{Aut} FT^{\widehat{\ }} \to \mathcal{E} \to S \to 1$$

of S by the prounipotent radical of $\operatorname{Aut} FT^{\widehat{\ }}$ by pulling back the extension

$$1 \to J^{-1}\operatorname{Aut} FT^{\widehat{\ }} \to \operatorname{Aut} FT^{\widehat{\ }} \to \operatorname{Aut} H_1(T; F)$$

along $S \to \operatorname{Aut} H_1(T; F)$. The representation $\Gamma \to \operatorname{Aut} FT^{\widehat{\ }}$ lifts to a representation $\Gamma \to \mathcal{E}$ whose composition with the projection $\mathcal{E} \to S$ is $\rho : \Gamma \to S$. This induces a homomorphism $\mathcal{G}_F \to \mathcal{E}$. Since the composite

$$\mathcal{T}_F \to \mathcal{G}_F \to \mathcal{E} \to \operatorname{Aut} FT^{\widehat{\ }}$$

is the action of $\mathcal{T}_F \subseteq FT^{\widehat{\ }}$ on $FT^{\widehat{\ }}$ by inner automorphisms, it follows that the kernel of this map is the center $Z(\mathcal{T}_F)$ of $\mathcal{T}_F$. It follows that the kernel of $\mathcal{T}_F \to \mathcal{U}_F$ is a subgroup of $Z(\mathcal{T})$. $\quad\square$

The following result and its proof were communicated to me by P. Deligne [5].

Proposition 4.6 *Suppose that S is semisimple and that the natural action of L on $H_1(T; F)$ extends to a rational representation of S. If $H^1(L; A) = 0$ for all rational representations A of S, then Φ is surjective.*

Proof. The homomorphism $\mathcal{T}_F \to \mathcal{U}_F$ is surjective if and only if the induced map $H_1(T; F) \to H_1(\mathcal{U}_F)$ is surjective. Let A be the cokernel of this map. This is a rational representation of S as both $H_1(T; F)$ and $H_1(\mathcal{U}_F)$ are. By pushing out the extension

$$1 \to \mathcal{U}_F \to \mathcal{G}_F \to S \to 1$$

along the map $\mathcal{U}_F \to H_1(U) \to A$, we obtain an extension

$$1 \to A \to G \to S \to 1$$

of algebraic groups and a homomorphism $\Gamma \to G$ which lifts ρ and has Zariski dense image in G. Let

$$1 \to A \to G^L \to L \to 1$$

be the restriction of this extension to L. Since A is the cokernel of the map $\mathcal{T}_F \to H_1(U)$, the image of Γ in G is $\Gamma/T = L$. So ρ induces a homomorphism $\tilde{\rho} : L \to G$ which has Zariski dense image. The image of $\tilde{\rho}$ lies in G^L and induces a splitting $\sigma : L \to G^L$ of the projection $G^L \to L$. Since G is an algebraic group, there is a splitting $s : S \to G$ of the projection in the category of algebraic groups. This restricts to a splitting $s' : L \to G$. Since $H^1(L; A)$ vanishes, there exists $a \in A$ which conjugates s' into σ. Thus the image of σ is

contained in the algebraic subgroup $as(S)a^{-1}$ of G. Since the image of σ in G is Zariski dense, it follows that A must be trivial. $\square$

A direct consequence of this result is a criterion for the map $\rho : \Gamma \to S$ itself to be the ρ completion. This criterion is satisfied by arithmetic groups in semisimple groups where each factor has $\mathbb{Q}$ rank ≥ 2 [23].

Corollary 4.7 *Suppose that $\rho : \Gamma \to S$ is a homomorphism of an abstract group into a semisimple algebraic group. If $H^1(L; A)$ vanishes for all rational representations of S, and if $H^1(T; F)$ is zero (e.g., ρ injective), then the relative completion of Γ with respect to ρ is $\rho : \Gamma \to S$.* $\square$

Next we consider the problem of imbedding an extension of L by a unipotent group U in an extension of S by U.

Proposition 4.8 *Suppose that*

$$1 \to U \to G \to L \to 1$$

is a split extension of abstract groups, where U is a unipotent group over F and where the action of L on $H^1(U)$ extends to a rational representation of S. If $H^1(L; A)$ vanishes for all rational representations of S, then there is an extension

$$1 \to U \to \tilde{G} \to S \to 1$$

of algebraic groups such that the original extension is the restriction of this one to L.

Proof. Since the first extension splits, we can write it as a semi direct product

$$G = L \ltimes U.$$

Denote the Lie algebra of U by $\mathfrak{u}$. The group of Lie algebra automorphisms of $\mathfrak{u}$ is an algebraic group over F. It can be expressed as an extension

$$1 \to J^{-1}\mathrm{Aut}\,\mathfrak{u} \to \mathrm{Aut}\,\mathfrak{u} \to \mathrm{Aut}\,H_1(\mathfrak{u})$$

where the kernel consists of those automorphisms which act trivially on $H^1(\mathfrak{u})$, and therefore on the graded quotients of the lower central series of $\mathfrak{u}$. It is a unipotent group. We can pull this extension back along the representation $S \to \mathrm{Aut}\,H_1(\mathfrak{u})$ to obtain an extension

$$1 \to J^{-1}\mathrm{Aut}\,\mathfrak{u} \to \tilde{A} \to S \to 1$$

The representation $L \to \mathrm{Aut}\,\mathfrak{u}$ lifts to a homomorphism $L \to \tilde{A}$. This induces a homomorphism of the S-completion of L into $\tilde{A}$. By (4.7), the completion of L is simply the inclusion $L \hookrightarrow S$. That is, the representation $L \to \tilde{A}$ extends to an

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algebraic group homomorphism $S \to \tilde{A}$. This implies that the representation of L on $\mathfrak{u}$ extends to a rational representation $S \to \text{Aut}\,\mathfrak{u}$. We can therefore form the semi direct product $S \ltimes U$, which is an algebraic group. The homomorphism $L \ltimes U \to S \ltimes U$ exists because of the compatibility of the actions of L and S on U. $\quad\square$

Combining this with (4.2), we obtain:

Corollary 4.9 *Suppose that $\mathcal{U}$ is a prounipotent group over F with $H_1(\mathcal{U})$ finite dimensional, and suppose that $\mathcal{Z}$ is a central subgroup of $\mathcal{U}$. Suppose that*

$$1 \to \mathcal{U} \to G \to L \to 1 \tag{4}$$

is an extension of abstract groups where the action of L on $H_1(\mathcal{U})$ extends to a rational representation of S. Suppose that

$$1 \to \mathcal{U}/\mathcal{Z} \to \mathcal{G} \to S \to 1$$

is an extension of proalgebraic groups which gives the extension

$$1 \to \mathcal{U}/\mathcal{Z} \to G/\mathcal{Z} \to L \to 1$$

when restricted to L. If the class in $H^2(L; Z)$ given by (4.1) vanishes, then there exists an extension

$$1 \to \mathcal{U} \to \tilde{\mathcal{G}} \to S \to 1$$

of proalgebraic groups whose restriction to L is the extension (4). $\quad\square$

By pushing out the extension

$$1 \to T \to \Gamma \to L \to 1$$

along the homomorphism $T \to \mathcal{T}_F$, we obtain a "fattening" $\hat{\Gamma}$ of Γ. Let $\mathcal{G}^L$ be the inverse image of L in $\mathcal{G}_F$. Using the universal mapping property of pushouts, one can show easily that the natural homomorphism $\Gamma \to \mathcal{G}_F$ induces homomorphism $\hat{\Gamma} \to \mathcal{G}^L$. These groups fit into a commutative diagram of extensions:

$$
\begin{array}{ccccccccc}
1 & \to & T & \to & \Gamma & \to & L & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \mathcal{T}_F & \to & \hat{\Gamma} & \to & L & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \mathcal{U}_F & \to & \mathcal{G}^L & \to & L & \to & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \to & \mathcal{U}_F & \to & \mathcal{G}_F & \to & S & \to & 1
\end{array}
$$

Next we introduce conditions we need to impose on our extension for the remainder of the section.

(**4.10**) We will now assume that the extension Γ of L by T satisfies the following conditions. First, $H_1(T; F)$ is finite dimensional and the action of L on it extends to a rational representation of S. Next, we assume that $H^1(L; A)$ vanishes for all rational representations of S. Finally, we add the new condition that $H^2(L; A)$ vanishes for all *nontrivial* irreducible rational representations of S. These conditions are satisfied by arithmetic groups in semisimple groups where each factor has real rank at least 8 [1, 2].

The following result is an immediate consequence of (4.9).

Proposition 4.11 *If the conditions (4.10) are satisfied, then $\mathcal{K}_F = \ker \Phi$ is contained in the center of the thickening $\hat{\Gamma}$ of Γ.* $\square$

Applying the construction (4.1) to the thickening $\hat{\Gamma}$ of Γ and a splitting of $\mathcal{G}^L \to L$, we obtain an extension

$$0 \to \mathcal{K}_F \to G \to L \to 1 \tag{5}$$

which is unique up to isomorphism. It follows from (4.11) that this is a central extension. Since $H_1(L; F)$ vanishes, there is a universal central extension with kernel an F vector space. It is the extension

$$0 \to H_2(L; F) \to \tilde{L} \to L \to 1$$

with cocycle the identity map

$$\left\{ H_2(L; F) \xrightarrow{id} H_2(L; F) \right\} \in \mathrm{Hom}(H_2(L; F), H_2(L; F)) \approx H^2(L; H_2(L; F)).$$

The central extension (5) is classified by a linear map $\psi_F : H_2(L; F) \to \mathcal{K}_F$. Because all splittings $s : L \to \mathcal{G}^L$ are conjugate (4.2), the class of this extension is independent of the choice of the splitting.

Since $\mathcal{K}_F$ is an abelian unipotent group, $\mathcal{K}_F(k) = \mathcal{K}_F \otimes k$ for all fields k which contain F. The homomorphism ψ_F satisfies the following naturality property:

Proposition 4.12 *If k is an extension field of F, then the diagram*

$$
\begin{array}{ccc}
H_2(L; k) & \to & \mathcal{K}_k \\
\| & & \downarrow \\
H_2(L; F) \otimes k & \to & \mathcal{K}_F(k)
\end{array}
$$

commutes. $\square$

The next result bounds the size of $\mathcal{K}_F$.

Proposition 4.13 *If the conditions (4.10) hold, then the natural map $\psi_F : H_2(L; F) \to \mathcal{K}_F$ is surjective.*

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Proof. As above, we shall denote by G the central extension of L by $\mathcal{K}_F$. Let A be the cokernel of ψ_F and E the cokernel of $\psi_F : H_2(L; F) \to G$. Then E is a central extension of L by A. Because the composite $H_2(L; F) \to \mathcal{K}_F \to A$ is trivial, this extension is split. From (4.2) it follows that the extension

$$1 \to \mathcal{T}_F/\operatorname{im}\psi_F \to \hat{\Gamma}/\operatorname{im}\psi_F \to L \to 1$$

is split. By (4.8), this implies that there is a proalgebraic group $\mathcal{E}$ which is a semidirect product of S by $\mathcal{T}_F/\operatorname{im}\psi_F$ into which $\hat{\Gamma}/\operatorname{im}\psi_F$ injects. The map of Γ to $\mathcal{E}$ induces a map $\mathcal{G}_F \to \mathcal{E}$. Since the kernel of the map $\mathcal{T}_F \to \mathcal{E}$ is $\operatorname{im}\psi_F$, it follows that $\mathcal{K}_F$ is contained in $\operatorname{im}\psi_F$. $\quad\square$

Corollary 4.14 *If the conditions (4.10) hold, then the natural map $\mathcal{G}_k \to \mathcal{G}_F(k)$ associated to a field extension $k : F$ is an isomorphism.*

Proof. By (3.4), the natural map $\mathcal{T}_F \to \mathcal{T}_F(k)$ is an isomorphism. Since $\mathcal{T}_K \to \mathcal{U}_K$ is surjective for all fields K, the natural map $\mathcal{U}_k \to \mathcal{U}_F(k)$ is also surjective. Consequently, the natural map $K_k \to K_F(k)$ is injective. Since $\mathcal{K}_F$ is abelian unipotent, $\mathcal{K}_F(k) = \mathcal{K}_F \otimes k$. But it follows from (4.13) that $\mathcal{K}_k \to \mathcal{K}_F(k)$ is surjective, and therefore an isomorphism. $\quad\square$

5 The Johnson homomorphism

This section is a brief review of the construction of Johnson's homomorphism [19, 17]. There are two equivalent ways, both due to Johnson, to define a homomorphism

$$T_g \to \Lambda^3 H_1(C; \mathbb{Z})$$

where C is a compact Riemann surface of genus g.

Choose a base point x of C. The first construction uses the action of T_g^1 on $\pi_1(C, x)$. Denote the lower central series of $\pi_1(C, x)$ by

$$\pi_1(C, x) = \pi^1 \supseteq \pi^2 \supseteq \pi^3 \supset \cdots$$

The first graded quotient π^1/π^2 is $H_1(C; \mathbb{Z})$. The second is naturally isomorphic to $\Lambda^2 H_1(C; \mathbb{Z})/\langle q \rangle$, where $q : \Lambda^2 H^1(C; \mathbb{Z}) \to H^2(C; \mathbb{Z}) \approx \mathbb{Z}$ is the cup product. If γ_j, $j = 1, \ldots, 2g$, are generators of $\pi_1(C, x)$, then the residue class of the commutator $\gamma_j \gamma_k \gamma_j^{-1} \gamma_k^{-1}$ modulo π^3 is the element $c_j \wedge c_k$ of $\Lambda^2 H_1(C; \mathbb{Z})/\langle q \rangle$, where c_j denotes the homology class of γ_j. The form q is just the equivalence class of the standard relation in $\pi_1(C, x)$.

If $\phi : (C, x) \to (C, x)$ is a diffeomorphism which represents an element of T_g^1, then ϕ acts trivially on the homology of C. It therefore acts as the identity on each graded quotient of the lower central series of $\pi_1(C, x)$. Define a function $\pi \to \pi$ by taking γ to $\phi(\gamma)\gamma^{-1}$. Since ϕ acts trivially on $H_1(C)$, it follows that this map takes π^l into π^{l+1}. In particular, it induces a well defined function

$$\tilde{\tau}(\phi) : H_1(C; \mathbb{Z}) \to \Lambda^2 H_1(C; \mathbb{Z})/\langle q \rangle$$

between the first two graded quotients of π, which is easily seen to be linear. Using Poincaré duality, $\tilde{\tau}(\phi)$ can be regarded as an element of

$$H_1(C; \mathbb{Z}) \otimes \left(\Lambda^2 H_1(C; \mathbb{Z}) / \langle q \rangle \right).$$

The map $\phi \mapsto \tilde{\tau}(\phi)$ induces a group homomorphism

$$T_g^1 \to H_1(C; \mathbb{Z}) \otimes \left(\Lambda^2 H_1(C; \mathbb{Z}) / \langle q \rangle \right).$$

and therefore a homomorphism

$$\hat{\tau} : H_1(T_g^1) \to H_1(C; \mathbb{Z}) \otimes \left(\Lambda^2 H_1(C; \mathbb{Z}) / \langle q \rangle \right).$$

There is a natural inclusion

$$\Lambda^3 H_1(C; \mathbb{Z}) \to H_1(C; \mathbb{Z}) \otimes \left(\Lambda^2 H_1(C; \mathbb{Z}) / \langle q \rangle \right).$$

defined by

$$x \wedge y \wedge z \mapsto x \otimes (y \wedge z) + y \otimes (z \wedge x) + z \otimes (x \wedge y).$$

Johnson has proved that the image of $\hat{\tau}$ is contained in the image of this map, so that $\hat{\tau}$ induces a homomorphism

$$\tau_g^1 : H_1(T_g^1) \to \Lambda^3 H^1(C; \mathbb{Z}).$$

It is not difficult to check that this homomorphism is $Sp_g(\mathbb{Z})$ equivariant.

This story can be extended to T_g as follows. There is a natural extension

$$1 \to \pi_1(C, x) \to T_g^1 \to T_g \to 1.$$

Applying H_1, we obtain the diagram

$$
\begin{array}{ccccccc}
H_1(C; \mathbb{Z}) & \to & H_1(T_g^1) & \to & H_1(T_g) & \to & 0 \\
 & & \downarrow{\scriptstyle \tau_g^1} & & & & \\
 & & \Lambda^3 H_1(C; \mathbb{Z}) & & & &
\end{array}
$$

whose top row is exact. Identify the lower group with $H_3(\mathrm{Jac}\, C; \mathbb{Z})$. Since $\mathrm{Jac}\, C$ is a group with torsion free homology, the group multiplication induces a product on its homology which is called the *Pontrjagin product*. The composite of τ_g^1 with the map from $H_1(C)$ takes a class in $H_1(\mathrm{Jac}\, C)$ to its Pontrjagin product with $q = [C] \in H_2(\mathrm{Jac}\, C; \mathbb{Z})$. It follows that τ_g^1 induces a map

$$\tau_g : H_1(T_g) \to \Lambda^3 H_1(C; \mathbb{Z}) / \left(q \wedge H_1(C; \mathbb{Z}) \right).$$

The following fundamental theorem is due to Dennis Johnson.

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Theorem 5.1 *When $g \geq 3$, the homomorphisms τ_g^1 and τ_g are isomorphisms modulo 2 torsion.*

The second way to construct a map $T_g^1 \to \Lambda^3 H_1(C; \mathbb{Z})$ is as follows. Suppose that ϕ represents an element of T_g^1. Let $M_\phi \to S^1$ be the bundle over the circle constructed by identifying the point $(z, 1)$ of $C \times [0, 1]$ with the point $(\phi(z), 0)$. Since ϕ fixes the basepoint x, the map $t \to (x, t)$ induces a section of $M_\phi \to S^1$. Similarly, one can construct the bundle of jacobians; this is trivial as ϕ acts trivially on $H_1(C)$. One can imbed M_ϕ in this bundle of jacobians using this section of basepoints. Let p be the projection of the bundle of jacobians onto one of its fibers. Then one has the 3 cycle $p_* M_\phi$ in $\mathrm{Jac}\, C$.

Proposition 5.2 [19] *When $g \geq 3$, the homology class*

$$p_*[M_\phi] \in H_3(\mathrm{Jac}\, C; \mathbb{Z}) \approx \Lambda^3 H_1(C; \mathbb{Z})$$

is $\tau_g^1(\phi)$. $\square$

This can be proved, for example, by checking that both maps agree on what Johnson calls "bounding pair" maps. These maps generate the Torelli group when $g \geq 3$ [16].

6   The cycle $C - C^-$

In this section we relate the algebraic cycle $C - C^-$ to Johnson's homomorphism.

Let C be a compact Riemann surface. Denote its jacobian by $\mathrm{Jac}\, C$. This is defined to be $\mathrm{Pic}^0 C$, the group of divisors of degree zero on C modulo principal divisors. Each divisor D of degree zero may be written as the boundary of a topological 1-chain: $D = \partial \gamma$. Taking D to the functional $\omega \mapsto \int_\gamma \omega$ on the space $\Omega(C)$ of holomorphic 1-forms yields a well defined map

$$\mathrm{Pic}^0 C \to \Omega(C)^* / H_1(C; \mathbb{Z}). \tag{6}$$

This is an isomorphism by Abel's Theorem.

For each $x \in C$, we have an Abel-Jacobi map

$$\nu_x : C \to \mathrm{Jac}\, C$$

which is defined by $\nu_x(y) = y - x$. This map is an imbedding if the genus g of C is ≥ 1. Denote its image by C_x. This is an algebraic 1-cycle in $\mathrm{Jac}\, C$. Denote the involution $D \mapsto -D$ of $\mathrm{Jac}\, C$ by i, and the image of C_x under this involution by C_x^-. Since i induces $-id$ on $H^1(\mathrm{Jac}\, C; \mathbb{Z})$, and since i^* is a ring homomorphism, we see that i^* acts as $(-1)^k$ on $H_k(\mathrm{Jac}\, C)$. It follows that C_x and C_x^- are homologically equivalent. Set $Z_x = C_x - C_x^-$. This is a homologically trivial 1-cycle.

Griffiths has a construction which associates to a homologically trivial analytic cycle in a compact Kähler manifold a point in a complex torus. His construction is a generalization the construction of the map (6). We review this construction briefly. Suppose that X is a compact Kähler manifold, and that Z is an analytic k-cycle in X which is homologous to 0. Write $Z = \partial\Gamma$, where Γ is a topological $2k + 1$ chain in X. Define

$$F^p H^m(X) = \bigoplus_{s \geq p} H^{s,m-s}(X).$$

Each class in $F^p H^m(X)$ can be represented by a closed form where each term of a local expression in terms of local holomorphic coordinates $(z_1, \ldots, z_n)$ has at least $p\ dz_j$s. Integrating such representatives of classes in $F^{k+1} H^{2k+1}(X)$ over Γ gives a well defined functional

$$\int_\Gamma : F^{k+1} H^{2k+1}(X) \to \mathbb{C}.$$

The choice of Γ is unique up to a topological $2k + 1$ cycle. So Z determines a point of the complex torus

$$J_k(X) := F^{k+1} H^{2k+1}(X)^* / H_{2k+1}(X; \mathbb{Z}).$$

This group is called the *kth intermediate jacobian of X*.

In our case, the cycle Z_x determines a point $\zeta_x(C)$ in the intermediate jacobian

$$J_1(\operatorname{Jac} C) = F^2 H^3(\operatorname{Jac} C)^* / H_3(\operatorname{Jac} C; \mathbb{Z}).$$

The homology class of C_x is easily seen to be independent of x. Taking the Pontrjagin product with $[C]$ defines an injective map

$$H_1(\operatorname{Jac} C; \mathbb{Z}) \hookrightarrow H_3(\operatorname{Jac} C; \mathbb{Z}).$$

The dual $H^3(\operatorname{Jac} C) \to H^1(C)$ is a morphism of Hodge structures of type $(-1, -1)$ — that is, $H^{s,t}$ gets mapped into $H^{s-1,t-1}$. It follows that this map induces an imbedding

$$\Phi : \operatorname{Jac} C \hookrightarrow J_1(\operatorname{Jac}(C))$$

of complex tori. We shall denote the cokernel of this map by $JQ_1(\operatorname{Jac} C)$. It is trivial when $g < 3$.

The following result is not difficult; a proof may be found in [22].

Proposition 6.1 *If $x, y \in C$, then*

$$\zeta_x(C) - \zeta_y(C) = 2\Phi(x - y).$$

In particular, the image of $\zeta_x(C)$ in $JQ_1(\operatorname{Jac} C)$ is independent of x.

Denote the common image of the $\zeta_x(C)$ in $JQ_1(\operatorname{Jac} C)$ by $\zeta(C)$.

We now suppose that the genus g of C is ≥ 3. Fix a level l so that the moduli space $\mathcal{M}_g^n(l)$ of curves of genus g and n marked points with a level l structure is smooth. (Any $l \geq 3$ will do for the time being.) Denote the space of principally polarized abelian varieties of dimension g with a level l structure by $\mathcal{A}_g(l)$.

One can easily construct bundles $\mathcal{J}_m \rightarrow \mathcal{A}_g(l)$ of complex tori whose fiber over the abelian variety A is $J_m(A)$. The pullback of $\mathcal{J}_0$ along the period map $\mathcal{M}_g(l) \rightarrow \mathcal{A}_g(l)$ is the bundle of jacobians associated to the universal curve. The imbedding $\mathcal{J}_0 \hookrightarrow \mathcal{J}_1$ defined over $\mathcal{M}_g(l)$ extends to all of $\mathcal{A}_g(l)$ as the homology class $[C] \in H_2(\operatorname{Jac} C; \mathbb{Z})$ extends to a class $q \in H_2(A; \mathbb{Z})$ for every abelian variety A. Denote the bundle of quotient tori by $\mathcal{Q} \rightarrow \mathcal{A}_g(l)$.

Denote the level l congruence subgroup of $Sp_g(\mathbb{Z})$ by $L(l)$. Since $\mathcal{A}_g(l)$ is an Eilenberg-Mac Lane space $K(L(l), 1)$, and since the fiber of $\mathcal{J}_1 \rightarrow \mathcal{A}_g(l)$ is a $K(H_3(\operatorname{Jac} C; \mathbb{Z}), 1)$, it follows that $\mathcal{J}_1$ is also an Eilenberg-Mac Lane space whose fundamental group is an extension of $L(l)$ by $H_3(\operatorname{Jac} C; \mathbb{Z})$. Since this bundle has a section, viz., the zero section, this extension splits. The action of $L(l)$ on $H_3(\operatorname{Jac} C; \mathbb{Z})$ is the restriction of the third exterior power of the fundamental representation of Sp_g. There is a similar story with $\mathcal{J}_1$ replaced by $\mathcal{Q}$.

We shall denote the quotient $\Lambda^3 H_1(C; \mathbb{Z}) / ([C] \cdot H_1(C; \mathbb{Z}))$ by $Q\Lambda^3 H_1(C)$. This is the fundamental group of $JQ_1(\operatorname{Jac} C)$.

Proposition 6.2 *The spaces $\mathcal{J}_1$ and $\mathcal{Q}$ are Eilenberg-Mac Lane spaces with fundamental groups*

$$\pi_1(\mathcal{J}_1, 0_C) \approx L(l) \ltimes \Lambda^3 H_1(C; \mathbb{Z})$$

and

$$\pi_1(\mathcal{Q}, 0_C) \approx L(l) \ltimes Q\Lambda^3 H_1(C),$$

respectively. Here 0_C denotes the identity element in the fiber over $\operatorname{Jac} C$.

The normal functions $(C, x) \mapsto \zeta_x(C)$ and $C \mapsto \zeta(C)$ give lifts of the period map:

$$
\begin{array}{ccccccc}
 & & \mathcal{J}_1 & & & & \mathcal{Q} \\
 & \zeta_g^1 \nearrow & \downarrow & & & \zeta_g \nearrow & \downarrow \\
\mathcal{M}_g^1(l) & \rightarrow & \mathcal{A}_g(l) & & \mathcal{M}_g(l) & \rightarrow & \mathcal{A}_g(l)
\end{array}
$$

These induce maps of fundamental groups

$$\zeta_{g\,*}^1 : \Gamma_g^1(l) \rightarrow L(l) \ltimes \Lambda_3 H_1(C)$$

and

$$\zeta_{g\,*} : \Gamma_g(l) \rightarrow L(l) \ltimes Q\Lambda^3 H_1(C)$$

Since these maps commute with the canonical projections to $L(l)$, these induce $L(l)$ equivariant maps

$$\zeta_g^1 : H_1(T_g^1) \rightarrow \Lambda^3 H_1(C; Z), \quad \zeta_g^1 : H_1(T_g) \rightarrow Q\Lambda^3 H_1(C; Z)$$

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The following result follows easily from (5.2). The factor of 2 arises as both C and C^- each contribute a copy of the Johnson homomorphism.

Proposition 6.3 *The map ζ_g^n is twice Johnson's map τ_g^n for $n = 0, 1$.* $\qquad\square$

It is natural to try to give a "motivic description" of the Johnson homomorphism rather than of twice it. Looijenga (unpublished) has done this by constructing a normal function which compares the cycle C_x to a fixed topological (but not algebraic) cycle in $\mathrm{Jac}\, C$ which is homologous to C_x. At the cost of being more abstract, we give another description which does not make use of Looijenga's topological cycle.

We will only consider the pointed case, the unpointed case being similar. For each pointed curve (C, x), the cycle C_x determines a point c_x in the Deligne cohomology group $H_{\mathcal{D}}^{2g-2}(\mathrm{Jac}\,(C), \mathbb{Z}(g-1))$. This group is an extension of the Hodge classes $H_{\mathbb{Z}}^{g-1,g-1}(\mathrm{Jac}\, C)$ by the intermediate jacobian $J_1(\mathrm{Jac}\, C)$:

$$0 \to J_1(\mathrm{Jac}\, C) \to H_{\mathcal{D}}^{2g-2}(\mathrm{Jac}\,(C), \mathbb{Z}(g-1)) \to H_{\mathbb{Z}}^{g-1,g-1}(\mathrm{Jac}\, C) \to 0$$

One can consider the bundle over $\mathcal{A}_g(l)$ whose fiber over the abelian variety A is $H_{\mathcal{D}}^{2g-2}(A, \mathbb{Z}(g-1))$. The subbundle whose fiber over A is the $J_1(A)$ coset of the class of the polarization $q \in H_{\mathbb{Z}}^{g-1,g-1}(A)$ is a principal $\mathcal{J}_1$ torsor over $\mathcal{A}_g(l)$. Denote it by $\mathcal{Z} \to \mathcal{A}_g(l)$. The cycle gives a lift

$$
\begin{array}{ccc}
 & & \mathcal{Z} \\
 & c \nearrow & \downarrow \\
\mathcal{M}_g^1(l) & \to & \mathcal{A}_g(l)
\end{array}
$$

of the period map. The total space $\mathcal{Z}$ is an Eilenberg Mac Lane space whose fundamental group is an extension of $L(l)$ by $\Lambda^3 H_1(C; Z)$. The map induced by c on fundamental groups induces the Johnson homomorphism τ_g^1. This follows directly from (5.2).

7 Completion of mapping class groups

Denote the completion of the mapping class group $\Gamma_{g,r}^n$ with respect to the canonical representation $\Gamma_{g,r}^n \to Sp_g$ by $\mathcal{G}_{g,r}^n$ and its prounipotent radical by $\mathcal{U}_{g,r}^n$. Denote the Malcev completion of the Torelli group $T_{g,r}^n$ by $\mathcal{T}_{g,r}^n$ and the kernel of the natural homomorphism $\mathcal{T}_{g,r}^n \to \mathcal{U}_{g,r}^n$ by $\mathcal{K}_{g,r}^n$. These groups are all defined over $\mathbb{Q}$ by (4.14).

For all $g \geq 3$, and all arithmetic subgroups L of $Sp_g(\mathbb{Z})$, $H^1(L; A)$ vanishes for all rational representations A of Sp_g. By the results of Borel [1, 2], the hypotheses (4.10) are satisfied by all arithmetic subgroups L of $Sp_g(\mathbb{Z})$ and $H^2(L; \mathbb{Q})$ is one dimensional when $g \geq 8$. This, combined with (4.13) yields the following result.

Proposition 7.1 *For all $g \geq 3$ and all $n, r \geq 0$, the natural map $\mathcal{T}_{g,r}^n \to \mathcal{U}_{g,r}^n$ is surjective. When $g \geq 8$, the kernel $\mathcal{K}_{g,r}^n$ is either trivial or isomorphic to $\mathbb{Q}$.* $\square$

Our main result is:

Theorem 7.2 *For all $g \geq 3$ and all $r, n \geq 0$, the kernel $\mathcal{K}_{g,r}^n$ is non-trivial, so that $\mathcal{K}_{g,r}^n \approx \mathbb{Q}$ when $g \geq 8$.*

Let $\lambda_1, \ldots, \lambda_g$ be a fundamental set of weights of Sp_g. For a dominant integral weight λ, denote the irreducible representation with highest weight λ by $V(\lambda)$.

In this section we will reduce the proof of Theorem 7.2 to the proof of the the case $g = 3$ and $r = n = 0$. The following assertion follows easily by induction on n and r from Johnson's result.

Proposition 7.3 *If $g \geq 3$, then for all $n, r \geq 0$, there is an Sp_g equivariant isomorphism*

$$H_1(T_{g,r}^n; \mathbb{Q}) \approx V(\lambda_3) \oplus V(\lambda_1)^{n+r}. \quad \square$$

The important fact for us is that the multiplicity of λ_3 in $H_1(T_{g,r}^n; \mathbb{Q})$ is always 1. Denote the second graded quotient of the lower central series of $\mathcal{T}_{g,r}^n$ by $\mathcal{V}_{g,r}^n$. The commutator induces a linear surjection

$$\Lambda^2 H_1(T_{g,r}^n; \mathbb{Q}) \to \mathcal{V}_{g,r}^n$$

which is Sp_g equivariant. By Schur's lemma, there is a unique copy of the trivial representation in $\Lambda^2 V(\lambda_3)$. Let $\beta_{g,r}^n : \mathbb{Q} \to \mathcal{V}$ be the composite

$$\mathbb{Q} \hookrightarrow \Lambda^2 V(\lambda_3) \to \Lambda^2 H_1(T_{g,r}^n; \mathbb{Q}) \to \mathcal{V}_{g,r}^n,$$

where the first map is the inclusion of the trivial representation. Consider the following assertions:

$A_{g,r}^n$:   The map $\beta_{g,r}^n$ is injective.

Proposition 7.4 *If $h \geq g \geq 3$, $s \geq r \geq 0$ and $m \geq n \geq$, then $A_{g,r}^n$ implies $A_{h,s}^m$. Furthermore $A_{g,1}$ implies A_g. In particular, A_3 implies $A_{g,r}^n$ for all $g \geq 3$ and $n, r \geq 0$.*

Proof. It is easy to see that $\beta_{g,r}^n$ is the composite of $\beta_{g,s}^m$ with the canonical quotient map $\mathcal{V}_{g,s}^m \to \mathcal{V}_{g,r}^n$ whenever $m \geq n$ and $s \geq r$. So $A_{g,r}^n$ implies $A_{g,s}^m$. Moreover, when $r \geq 1$, the composition of $\beta_{g,r}^n$ with the map $\mathcal{V}_{g,r}^n \to \mathcal{V}_{g+1,r}^n$ induced by any one of the natural maps $\Gamma_{g,r}^n \to \Gamma_{g+1,r}^n$ is $\beta_{g+1,r}^n$. So, in this case, $A_{g,r}^n$ implies $A_{g+1,r}^n$.

To see that $A_{g,1}$ implies A_g, consider the group extension

$$1 \to \pi_1(T_1^*C, \vec{v}) \to T_{g,1} \to T_g \to 1$$

where T_1^*C denotes the unit cotangent bundle of C. It is not difficult to show that $\mathcal{V}_{g,1}$ is the direct sum of $\mathcal{V}_g$ and the second graded quotient of the lower central series of $\pi_1(T_1^*C, \vec{v})$, from which the assertion follows. $\quad\square$

The assertions $A_{g,r}^n$ can be proved using Harer's computations of $H^2(\Gamma_{g,r}^n; \mathbb{Q})$ [9, 10]. However, we will prove A_3 in the course establishing the rest of Theorem 7.2 directly, without appeal to Harer's computation.

Denote the fattening (see discussion following (4.2)) of the mapping class group $\Gamma_{g,r}^n$ by $\hat{\Gamma}_{g,r}^n$. This is an extension

$$1 \to \mathcal{T}_{g,r}^n \to \hat{\Gamma}_{g,r}^n \to Sp_g(\mathbb{Z}) \to 1.$$

Dividing out by the commutator subgroup of $\mathcal{T}_{g,r}^n$, we obtain an extension

$$0 \to H_1(T_{g,r}^n; \mathbb{Q}) \to \mathcal{E}_{g,r}^n \to Sp_g(\mathbb{Z}) \to 1. \tag{7}$$

This extension is split. This can be seen using a straightforward generalization of (6.3).

Now suppose $A_{g,r}^n$ holds. Then there is a quotient $G_{g,r}^n$ of $\mathcal{T}_{g,r}^n$ which is an extension of $H_1(T_{g,r}^n; \mathbb{Q})$ by $\mathbb{Q}$. The cocycle of the extension being a non-zero multiple of the polarization

$$\theta \in \Lambda^2 V(\lambda_3) \subseteq H^2(T_{g,r}^n; \mathbb{Q}).$$

Dividing $\hat{\Gamma}_{g,r}^n$ by the kernel of $\mathcal{T}_{g,r}^n \to G_{g,r}^n$, we obtain an extension

$$1 \to G_{g,r}^n \to E_{g,r}^n \to Sp_g(\mathbb{Z}) \to 1. \tag{8}$$

Since the extension (7) splits, we can apply the construction (4.1) to obtain an extension

$$0 \to \mathbb{Q} \to H_{g,r}^n \to Sp_g(\mathbb{Z}) \to 1$$

or equivalently, a class $e_{g,r}^n \in H^2(Sp_g(\mathbb{Z}); \mathbb{Q})$. By (4.2), the non-triviality of this class is the obstruction to splitting the extension (8), which, by (4.8) is the obstruction to imbedding it in an algebraic group extension of Sp_g by $G_{g,r}^n$. This proves the following statement.

Proposition 7.5 *If $A_{g,r}^n$ holds and if $e_{g,r}^n$ is non-zero, then $\mathcal{K}_{g,r}^n$ is non-trivial.*

To reduce the proof of Theorem (7.2) to the genus 3 case, we need to relate the classes $e_{g,r}^n$.

Proposition 7.6 *For fixed $g \geq 3$, the classes $e_{g,r}^n$ are all equal. The image of $e_{g+1,1}$ under the natural map $H^2(Sp_{g+1}(\mathbb{Z}); \mathbb{Q}) \to H^2(Sp_g(\mathbb{Z}); \mathbb{Q})$ is $e_{g,1}$.*

Proof. Both statements follow from the naturality of the construction. $\quad\square$

Combining (7.4) and (7.6), we have:

Proposition 7.7 *If A_3 holds and e_3 is non trivial, then Theorem 7.2 is true.*

8   Proof of Theorem 7.2

We prove Theorem 7.2 by proving (7.7). In this section we assume that the reader is familiar with mixed Hodge theory. We will use the notation and conventions of [8, §§2–3]. The moduli space $\mathcal{A}_g$ can be thought of as the moduli space of principally polarized Hodge structures of weight -1, level 1 and dimension $2g$; the abelian variety $A \in \mathcal{A}_g$ corresponds to the Hodge structure $H_1(A)$ and its natural polarization. We can construct bundles over $\mathcal{A}_g$ by considering moduli spaces of various mixed Hodge structures derived from such a Hodge structure of weight -1. To guarantee that we have a smooth moduli space, we fix a level l so that $\mathcal{A}_g(l)$ is smooth.

For a Hodge structure $H \in \mathcal{A}_g$, where $g \geq 3$, with principal polarization q, we define QH to be the Hodge structure

$$\left[\Lambda^3 H / (q \wedge H) \right] \otimes \mathbb{Z}(-1)$$

which is of weight -1. Denote the dual Hodge structure

$$\mathrm{Hom}(QH, \mathbb{Z}(1)) \approx \ker \left\{ \,_{\text{-}} \wedge q : \Lambda^{2g-3} H \to \Lambda^{2g-1} H \right\} \otimes \mathbb{Z}(2 - g)$$

by PH.

The set of all mixed Hodge structures with weight graded quotients $\mathbb{Z}$ and QH is naturally isomorphic to the complex torus

$$J(QH) := QH_{\mathbb{C}} / \left(F^0 QH_{\mathbb{C}} + QH_{\mathbb{Z}} \right).$$

If $H = H_1(\mathrm{Jac}\,C)$, then $J(QH)$ is the torus $JQ_1(\mathrm{Jac}\,C)$ defined in §6. As we let H vary over $\mathcal{A}_g(l)$, we obtain the bundle $\mathcal{Q} \to \mathcal{A}_g(l)$ of intermediate jacobians constructed in §6.

The set of all mixed Hodge structures with weight graded quotients QH and $\mathbb{Z}(1)$ is naturally isomorphic to the complex torus

$$J(PH) := PH_{\mathbb{C}} / \left(F^0 PH_{\mathbb{C}} + PH_{\mathbb{Z}} \right).$$

This torus is the dual of $J(QH)$. Performing this construction for each H in $\mathcal{A}_g(l)$, we obtain a bundle $\mathcal{P} \to \mathcal{A}_g(l)$ of complex tori.

We now construct a line bundle over the fibered product

$$\mathcal{P} \times_{\mathcal{A}_g(l)} \mathcal{Q} \to \mathcal{A}_g(l).$$

It is the *biextension line bundle*. For a Hodge structure $H \in \mathcal{A}_g$, let $B(H)$ be the set of mixed Hodge structure with weight graded quotients canonically isomorphic to $\mathbb{Z}$, QH, and $\mathbb{Z}(1)$. Set

$$G_{\mathbb{Z}} = \begin{pmatrix} 1 & QH_{\mathbb{Z}} & \mathbb{Z}(1) \\ 0 & 1 & PH_{\mathbb{Z}} \\ 0 & 0 & 1 \end{pmatrix} \quad G = \begin{pmatrix} 1 & QH_{\mathbb{C}} & \mathbb{C} \\ 0 & 1 & PH_{\mathbb{C}} \\ 0 & 0 & 1 \end{pmatrix}$$

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$$F^0 G_{\mathbb{Z}} = \begin{pmatrix} 1 & F^0 QH & 0 \\ 0 & 1 & F^0 PH \\ 0 & 0 & 1 \end{pmatrix}.$$

There is a natural isomorphism

$$B(H) = G_{\mathbb{Z}} \backslash G / F^0 G.$$

The natural projection

$$B(H) \to J(QH) \times J(PH),$$

which takes $V \in B(H)$ to $(V/W_{-2}, W_{-1}V)$, is a principal $\mathbb{C}^*$ bundle. Doing this construction over $\mathcal{A}_g(l)$, we obtain a $\mathbb{C}^*$ bundle

$$\mathcal{B} \to \mathcal{Q} \times_{\mathcal{A}_g(l)} \mathcal{P}$$

Denote the corresponding line bundle by $\mathcal{L} \to \mathcal{Q} \times_{\mathcal{A}_g(l)} \mathcal{P}$.

Since the fiber of $\mathcal{B} \to \mathcal{A}_g(l)$ over H is the nilmanifold $B(H)$, which is an Eilenberg-Mac Lane space, it follows that $\mathcal{B}$ is also an Eilenberg-Mac Lane space whose fundamental group of $\mathcal{B}$ is an extension

$$1 \to G_{\mathbb{Z}} \to \pi_1(\mathcal{B}, *) \to L(l) \to 1. \tag{9}$$

Taking $H \in \mathcal{A}_g(l)$ to the split biextension $\mathbb{Z} \oplus H \oplus \mathbb{Z}(1)$ defines a section of the bundle $\mathcal{B} \to \mathcal{A}_g$. It follows that the extension (9) is split.

Proposition 8.1 $\pi_1(\mathcal{B}, *) \approx L(l) \ltimes G_{\mathbb{Z}}$. $\quad\square$

We now restrict ourselves to genus 3 and consider the problem of lifting the period map $\mathcal{M}_3(l) \to \mathcal{A}_3(l)$ to $\mathcal{B}$. The idea is that such a lifting will be the period map of a variation of mixed Hodge structure. To this end we define certain algebraic cycles which are just more canonical versions of the cycle $C - C^-$ considered in §6.

Suppose that C is a curve of genus 3 and that α is a theta characteristic of C — i.e., α is a square root of the canonical divisor κ_C. Denote the algebraic cycle corresponding to the canonical inclusion $C \hookrightarrow \mathrm{Pic}^1 C$ by C. Denote the involution $x \mapsto \alpha - x$ of $\mathrm{Pic}^1 C$ by i^α. For $D \in \mathrm{Pic}^0 C$, denote the translation map $x \mapsto x + D$ by

$$\tau^D : \mathrm{Pic}^1 C \to \mathrm{Pic}^1 C.$$

For $D \in \mathrm{Pic}^0 C$, define $C_D = \tau_*^D C$ and $Z_{\alpha,D} = C_D - i_*^\alpha C_D$. Each $Z_{\alpha,D}$ is homologous to zero. Set $Z_\alpha = Z_{\alpha,0}$.

Let Θ_α be the *theta divisor*

$$\{x + y - \alpha : x, y \in C\} \subseteq \mathrm{Pic}^0 C$$

and Δ be the *difference divisor*

$$\{x - y : x, y \in C\} \subseteq \mathrm{Pic}^0 C.$$

The following fact is easily verified.

Proposition 8.2 *The cycles Z_α and $Z_{\alpha,D}$ have disjoint supports if and only if $D \notin \Theta_\alpha \cup \Delta$.* $\quad\square$

Now choose a point $\delta \in \mathrm{Pic}^0 C$ of order 2 such that $\beta := \alpha + \delta$ is an *even* theta characteristic. (i.e., $h^0(C, \beta) = 0$ or 2.)

Proposition 8.3 *The cycles Z_α and $Z_{\alpha,\delta}$ have disjoint supports except when C is hyperelliptic and either δ is the difference between two distinct Weierstrass points or $\alpha + \delta$ is the hyperelliptic series.*

Proof. By (8.2), Z_α and $Z_{\alpha,\delta}$ intersect if and only if $\delta \in \Delta$ or $\delta \in \vartheta_\alpha$. In the first case, there exist $x, y \in C$ such that $x - y = \delta \neq 0$. So $2x - 2y = 0$, which implies that C is hyperelliptic and that x and y are distinct Weierstrass points. In the second case, there are $x, y \in C$ such that

$$x + y = \alpha + \delta,$$

which implies that $\alpha + \delta$ is an effective theta characteristic. Since $\alpha + \delta$ is even by assumption, $h^0(C, \alpha + \delta) = 2$, which implies that C is hyperelliptic and that $\alpha + \delta$ is the hyperelliptic series. $\quad\square$

Next we use these cycles to construct various variations of mixed Hodge structure whose period maps give lifts of the period map $\mathcal{M}_3(l) \to \mathcal{A}_3(l)$ to $\mathcal{Q} \times_{\mathcal{A}_3(l)} \mathcal{P}$, and to $\mathcal{B}$ generically.

For each $D \in \mathrm{Pic}^0 C$, one has the extension of mixed Hodge structure

$$0 \to H_3(\mathrm{Pic}^1 C; \mathbb{Z}(-1)) \to H_3(\mathrm{Pic}^1 C, Z_{\alpha,D}; \mathbb{Z}(-1)) \to \mathbb{Z} \to 0$$

where the generator 1 of $\mathbb{Z}$ is the image of any relative class $[\Gamma]$ with $\partial\Gamma = Z_{\alpha,D}$. Pushing this extension out along the projection

$$H_3(\mathrm{Pic}^1 C) \to QH_3(\mathrm{Pic}^1 C)$$

one obtains an extension

$$0 \to QH_3(\mathrm{Pic}^1 C; \mathbb{Z}(-1)) \to E \to \mathbb{Z} \to 0. \tag{10}$$

This extension determines the same point in $J(QH_3(\mathrm{Pic}^1 C)) \approx Q_1(\mathrm{Jac}\, C)$ as the cycle $C_x - C_x^-$, and is independent of the choice of D.

Dually, one can consider the extension

$$0 \to \mathbb{Z}(1) \to H_3(\mathrm{Pic}^1 C - Z_{\alpha,D}; \mathbb{Z}(-1)) \to H_3(\mathrm{Pic}^1 C; \mathbb{Z}(-1)) \to 0 \tag{11}$$

which comes from the Gysin sequence. Here the canonical generator of $\mathbb{Z}(1)$ is the boundary of any 4 ball which is transverse to $Z_{\alpha,D}$ and has intersection number 1 with it.

Proposition 8.4 *If D is a point of order 2 in $\operatorname{Jac} C$, then there is a morphism of Hodge structures*

$$H_1(C, \mathbb{Z}) \to H_3(\operatorname{Pic}^1 C - Z_{\alpha,D}; \mathbb{Z}(-1))$$

whose composition with the natural map

$$H_3(\operatorname{Pic}^1 C - Z_{\alpha,D}; \mathbb{Z}(-1)) \to H_3(\operatorname{Pic}^1 C; \mathbb{Z}(-1))$$

is the map $\_ \times [C]$ given by Pontrjagin product with $[C]$.

Proof. The exact sequence of Hodge structures

$$0 \to H_1(C; \mathbb{Z}) \xrightarrow{\times [C]} H_3(\operatorname{Pic}^1 C; \mathbb{Z}(-1)) \to QH_3(\operatorname{Pic}^1 C; \mathbb{Z}(-1)) \to 0$$

induces an exact sequence of Ext groups

$$0 \to \operatorname{Ext}^1(QH_3(\operatorname{Pic}^1 C, \mathbb{Z}(-1)), \mathbb{Z}(1)) \to$$

$$\operatorname{Ext}^1(H_3(\operatorname{Pic}^1 C, \mathbb{Z}(-1)), \mathbb{Z}(1)) \to \operatorname{Ext}^1(H_1(C; \mathbb{Z}), \mathbb{Z}(1)) \to 0.$$

This sequence may be identified naturally with the sequence

$$0 \to JPH_3(\operatorname{Pic}^1 C; \mathbb{Z}(-1)) \to JH_3(\operatorname{Pic}^1 C; \mathbb{Z}(-1)) \xrightarrow{\phi} \operatorname{Jac} C \to 0,$$

where ϕ is the map induced by the morphism of Hodge structures $H_3(\operatorname{Jac} C; \mathbb{Z}) \to H_1(C; \mathbb{Z})$ defined by

$$x \times y \times z \mapsto (x \cdot y)z + (y \cdot z)x + (z \cdot x)y.$$

Here x, y, z are elements of $H_1(C; \mathbb{Z})$, $\times$ denotes the Pontrjagin product, and $(\ \cdot\ )$ denotes the intersection pairing. To prove the assertion, it suffices to show that the image in $\operatorname{Jac} C$ of the class of the extension (11) vanishes. Lefschetz duality gives an isomorphism of (11) with the the extension

$$0 \to H_3(\operatorname{Pic}^1 C; \mathbb{Z}(-1)) \to H_3(\operatorname{Pic}^1 C, Z_{\alpha,D}; \mathbb{Z}(-1)) \to \mathbb{Z} \to 0$$

It follows directly from [22, (6.7)] that the image of this extension under the map ϕ is $K_C - 2(\alpha + D) = 0$. $\square$

Taking $D = 0$ and dividing out by this copy of $H_1(C; \mathbb{Z})$, we obtain an extension

$$0 \to \mathbb{Z}(1) \to F \to QH_3(\operatorname{Pic}^1 C; \mathbb{Z}(-1)) \to 0. \tag{12}$$

Let $\mathcal{M}_3(l, \alpha, \delta)$ be the moduli space of genus 3 curves with a level l structure, a distinguished even theta characteristic α, and a distinguished point $\delta \in \operatorname{Pic}^0 C$ of order 2 such that $\alpha + \delta$ is also an even theta characteristic. Denote the universal jacobian over $\mathcal{M}_3(l, \alpha, \delta)$ by $\mathcal{J} \to \mathcal{M}_3(l, \alpha, \delta)$. The period maps for

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the extensions (10), (12), respectively, define maps $\mathcal{J} \to \mathcal{Q}$ and $\mathcal{J} \to \mathcal{P}$. These induce a map ϕ into their fibered product over $\mathcal{A}_3(l)$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{J} & \xrightarrow{\phi} & \mathcal{Q} \times_{\mathcal{A}_3(l)} \mathcal{P} \\
\downarrow & & \downarrow \\
\mathcal{M}_3(l, \alpha, \delta) & \to & \mathcal{A}_3(l)
\end{array}
$$

commutes. Pulling back the $\mathbb{C}^*$ bundle $\mathcal{B} \to \mathcal{Q} \times_{\mathcal{A}_3(l)} \mathcal{P}$ along ϕ gives a $\mathbb{C}^*$ bundle $\mathcal{L}^* \to \mathcal{J}$. Denote the corresponding line bundle by $\mathcal{L}$.

Denote the relative difference divisor in $\mathcal{J}$ by $\mathcal{D}$ and the relative theta divisor associated to α by ϑ_α

Lemma 8.5 *The Chern class of this line bundle is the divisor $\mathcal{J}$ is $2\mathcal{D} - 4\vartheta_\alpha$.*

Proof. We construct a meromorphic section of $\mathcal{L}$. A point of $\mathcal{J}$ is a curve C and a point D of $\operatorname{Jac} C$. According to (8.2), the cycles Z_α and $Z_{\alpha,D}$ have disjoint supports when $D \notin \Delta \cup \Theta_\alpha$. In this case we can consider the mixed Hodge structure $H_3(\operatorname{Pic}^1 C - Z_\alpha, Z_{\alpha,D}; \mathbb{Z}(-1))$.[4] Dividing this biextension out by the image of the composite

$$
H_1(C, \mathbb{Z}) \to H_3(\operatorname{Pic}^1 C - Z_\alpha; \mathbb{Z}(-1)) \to H_3(\operatorname{Pic}^1 C - Z_\alpha, Z_{\alpha,D}; \mathbb{Z}(-1))
$$

of the map of (11) with the natural inclusion produces a biextension $b_{C,\alpha}$ with weight graded quotients canonically isomorphic to

$$
\mathbb{Z}, \quad Q H_3(\operatorname{Jac} C; \mathbb{Z}(-1)), \quad \mathbb{Z}(1);
$$

the generator 1 of $\mathbb{Z}$ corresponding to any Γ with $\partial\Gamma = Z_{\alpha,D}$, and the canonical generator $2\pi i$ of $\mathbb{Z}(1)$ being the class of the boundary of any small 4 ball which is transverse to Z_α and having intersection number 1 with it. This defines a lift

$$
\tilde{\phi} : \mathcal{J} - (\mathcal{D} \cup \vartheta_\alpha) \to \mathcal{B}
$$

of the map $\phi : \mathcal{J} \to \mathcal{Q} \times_{\mathcal{A}_3(l)} \mathcal{P}$. It therefore defines a nowhere vanishing holomorphic section s of $\mathcal{L} \to \mathcal{J}$ on the complement of $\mathcal{D} \cup \vartheta_\alpha$. It follows from [8, (3.4.3)] that s extends to a meromorphic section of $\mathcal{L}$ on all of $\mathcal{J}$. Consequently, the Chern class of this bundle is supported on the divisor $\mathcal{D} \cup \vartheta_\alpha$. Since the divisors $\mathcal{D}$ and ϑ_α are irreducible, the Chern class can be computed by restricting to a general enough fiber.

With the aid of (8.4), and the formula [8, (3.2.11)], one can easily show that the height of the biextension $b_{C,\alpha}$ equals that of

$$
H_3(\operatorname{Pic}^1 C - Z_\alpha, \mathbb{Z}_{\alpha,D}; \mathbb{Z}(-1)).
$$

---

[4]This is everywhere a local system. One has to replace it with another group when C is hyperelliptic and either α or $\alpha + \delta$ is the hyperelliptic series. For details, see the footnote on page 887 of [8].

It follows from the main theorem of [8] that the divisor of s restricted to $\operatorname{Jac} C$ is $2D - 4\Theta_\alpha$ for all C. The result follows. $\quad\square$

The point δ of order 2 is a section of the bundle $\mathcal{J} \to \mathcal{M}_3(l, \alpha, \delta)$. A lift $\zeta : \mathcal{M}_3(l, \alpha, \delta) \to \mathcal{Q} \times_{\mathcal{A}_3(l)} \mathcal{P}$ of the period map can be defined by composing δ with ϕ. The pullback of the line bundle $\mathcal{L}$ along δ equals the pullback of the biextension line bundle to $\mathcal{M}_3$. It follows that the Chern class of this line bundle is $2\delta^*(\mathcal{D} - 2\vartheta_\alpha)$.

Proposition 8.6 *If α and $\alpha + \delta$ are even theta characteristics, then the push-forward of the divisor $\delta^*(\mathcal{D} - 2\vartheta_\alpha)$ in $\mathcal{M}_3(l, \alpha, \delta)$ to $\mathcal{M}_3(l)$ is 28.35 times the hyperelliptic locus. Consequently, the line bundle*

$$\delta^*\mathcal{L} \in \operatorname{Pic}\mathcal{M}_3(l, \alpha, \delta) \otimes \mathbb{Q}$$

is non-trivial.

In the proof of this result, we will need the following fact.

Lemma 8.7 *The section $\delta : \mathcal{M}_3(l, \alpha, \delta) \to \mathcal{J}$ is transverse to the divisors $\mathcal{D}$ and ϑ_α.*

Proof. We first prove that δ intersects ϑ_α transversally. We view $\mathcal{M}_3(l, \alpha, \delta)$ as a subvariety of $\mathcal{J}$ via the section δ. Let (C, α, δ) be a point in ϑ_α. Then, by (8.3), C is hyperelliptic, and $\alpha + \delta$ is the hyperelliptic series. So $h^0(\alpha) = 0$ and $h^0(\alpha + \delta) = 2$. Let Z_0 be the period matrix of C with respect to some symplectic basis of $H_1(C; \mathbb{Z})$, and $\theta_\alpha(u, Z)$ the theta function which defines ϑ_α. The point δ of order 2 may be viewed as a function $\delta(Z)$. Set $\delta_0 = \delta(Z_0)$. Since $\delta_0 \in \Theta_\alpha$, $\theta_\alpha(\delta_0, Z_0) = 0$. We have to show that there exist a, b such that

$$\frac{\partial}{\partial Z_{ab}}\theta_\alpha(\delta(Z), Z)|_{Z_0} \neq 0.$$

By Riemann's Theorem [6, p. 348], the multiplicity of δ on Θ_α is $h^0(\alpha + \delta) = 2$. That is,

$$\frac{\partial\theta_\alpha}{\partial u_a}(\delta_0, Z_0) = 0 \tag{13}$$

for all indices a, but there exist indices a, b such that

$$\frac{\partial^2\theta_\alpha}{\partial u_a \partial u_b}(\delta_0, Z_0) \neq 0.$$

Substituting (13) into the chain rule, we have

$$\frac{\partial\theta_\alpha}{\partial Z_{ab}}(\delta(Z), Z)|_{Z_0} = \sum_{j=1}^{g} \frac{\partial\theta_\alpha}{\partial u_j}(\delta_0, Z_0)\frac{\partial u_j}{\partial Z_{ab}}(\delta_0, Z_0) + \frac{\partial\theta_\alpha}{\partial Z_{ab}}(\delta_0, Z_0) = \frac{\partial\theta_\alpha}{\partial Z_{ab}}(\delta_0, Z_0).$$

Plugging this into the heat equation, we obtain the desired result:

$$\frac{\partial \theta_\alpha}{\partial Z_{ab}}(\delta(Z), Z)|_{Z_0} = \frac{\partial \theta_\alpha}{\partial Z_{ab}}(\delta_0, Z_0) = 2\pi i(1 + \delta_{ab})\frac{\partial^2 \theta_\alpha}{\partial u_a \partial u_b}(\delta_0, Z_0) \neq 0.$$

To prove that δ is transverse to $\mathcal{D}$, we use an argument suggested to us by Nick Katz. Consider a family of curves $C \to \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2)$ over the dual numbers. There is a relative difference divisor $\Delta \to \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2)$ contained in the Picard scheme $\mathrm{Pic}^0 C \to \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2)$. Suppose that we have a point of order 2

$$\delta : \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2) \to \mathrm{Pic}^0 C,$$

defined over the dual numbers which lies in Δ. To prove transversality, it suffices to show that C is hyperelliptic over the dual numbers. But this is immediate as δ gives a 2:1 map $C \to \mathbb{P}^1$ defined over the dual numbers. $\quad\square$

Proof of (8.6). Denote the hyperelliptic series of a hyperelliptic curve by H. It follows from (8.5) and (8.7) that

$$\delta^* \mathcal{D} = \mathcal{H}_\Delta \quad \text{and} \quad \delta^* \vartheta_\alpha = \mathcal{H}_0$$

where

$$\mathcal{H}_\Delta = \left\{(C, \alpha, \delta) : C \text{ is hyperelliptic }, \delta \in \Delta \text{ and } h^0(C, \alpha + \delta) \text{ is even}\right\}$$

and

$$\mathcal{H}_0 = \{(C, \alpha, \delta) : C \text{ is hyperelliptic }, \alpha \neq H, \alpha + \delta = H\}.$$

If C is hyperelliptic and if $\delta = x - y \in \Delta$, then x and y are distinct Weierstrass points, and

$$H + \delta = 2y + x - y = x + y$$

which is an odd theta characteristic. It follows that

$$\mathcal{H}_\Delta = \left\{(C, \alpha, \delta) : C \text{ is hyperelliptic }, \alpha \neq H, \delta \in \Delta \text{ and } h^0(C, \alpha + \delta) \text{ even}\right\}.$$

Apart from the hyperelliptic series, every even theta characteristic on a hyperelliptic curve C is of the form

$$-H + p_1 + p_2 + p_3 + p_4 = -H + q_1 + q_2 + q_3 + q_4$$

where $p_1, \ldots, p_4, q_1, \ldots, q_4$ are the Weierstrass points. If $\alpha = -H + p_1 + p_2 + p_3 + p_4$ and $\alpha + x - y$ is an even theta characteristic, then it is not difficult to show that $x - y = q_i - p_j$ for some i, j. So, for each even theta characteristic $\alpha \neq H$, there are 16 points $\delta \in \Delta$ such that $\alpha + \delta$ is also an even theta characteristic. Since there are 35 even theta characteristics $\alpha \neq H$, it follows that $\mathcal{H}_\Delta$ has degree 16.35 over the hyperelliptic locus $\mathcal{H}$ of $\mathcal{M}_3(l)$. Since there is only one point δ of order 2 such that $\alpha + \delta = H$, $\mathcal{H}_0$ has degree 35 over $\mathcal{H}$.

Putting this together we see that the pushforward of $\delta^*\mathcal{L}$ to $\mathcal{M}_3$ is

$$\pi_*(2\mathcal{H}_\Delta - 4\mathcal{H}_0) = (2.16.35 - 4.35)\mathcal{H} = 28.35\mathcal{H}. \quad \square$$

We are now ready to prove (7.7). Denote the subgroup of Γ_3 which corresponds to $\mathcal{M}_3(l, \alpha, \delta)$ by $\Gamma_3(l, \alpha, \delta)$, and its intersection with T_3 by $T_3(\alpha, \delta)$.

Proof of (7.7). Let N be the line bundle over $\mathcal{A}_3(l)$ which is the determinant of $R^1 f_*\mathcal{O}$, where $f : \mathcal{J} \to \mathcal{A}_3(l)$ is the universal abelian variety. The restriction of this to $\mathcal{M}_3(l)$ is $\mathcal{H}/9$ [12, p. 134]. We shall also denote its pullback to $\mathcal{Q} \times_{\mathcal{A}_3(l)} \mathcal{P}$ by N. It follows from (8.6) that the line bundle $\mathcal{L} \otimes N^{\otimes(-9.28.35)}$ pulls back to the trivial line bundle over $\mathcal{M}_3(l, \alpha, \delta)$. There is therefore a lift of the period map

$$\mathcal{M}_3(l, \alpha, \delta) \to \left(\mathcal{L} \otimes N^{\otimes(-9.28.35)}\right)^*$$

to the $\mathbb{C}^*$ bundle associated to $\mathcal{L} \otimes N^{\otimes(-9.28.35)}$. This induces a group homomorphism

$$\Gamma_3(l, \alpha, \delta) \to \pi_1(\left(\mathcal{L} \otimes N^{\otimes(-9.28.35)}\right)^*, *).$$

This last group is an extension

$$1 \to G_{\mathbb{Z}} \to \pi_1(\left(\mathcal{L} \otimes N^{\otimes(-9.28.35)}\right)^*, *) \to L(l) \to 1.$$

It follows from (6.3) and the fact that $T_3(\alpha, \delta)$ has finite index in T_3 that the image of the map

$$H_1(T_3(\alpha, \delta); \mathbb{Q}) \to H_1(G_{\mathbb{Z}}; \mathbb{Q}) = QH_3(\mathrm{Jac}\, C; \mathbb{Q}) \oplus PH_3(\mathrm{Jac}\, C; \mathbb{Q}) \approx V(\lambda_3)^2$$

is the diagonal copy of $V(\lambda_3)$. The restriction of the extension

$$1 \to \mathbb{Q} \to G_{\mathbb{Q}} \to V(\lambda^3)^2 \to 1$$

to the the diagonal is the extension given by the polarization of $V(\lambda_3)$. It follows that the homomorphism

$$\mathbb{Q} \hookrightarrow \Lambda^2 H_1(T_3; \mathbb{Q}) \to \mathbb{Q}$$

given by evaluating the bracket on the polarization is an isomorphism, as claimed. Second, the element of $H^2(L(l); \mathbb{Q})$ which corresponds to the extension

$$0 \to \mathbb{Q} \to H \to L(l) \to 1$$

constructed from the Torelli group is just the Chern class of the pullback of the line bundle $\mathcal{L} \otimes N^{\otimes(-9.28.35)}$ pulled back to $\mathcal{A}_3(l)$ along the zero section of $\mathcal{Q} \times_{\mathcal{A}_3(l)} \mathcal{P} \to \mathcal{A}_3(l)$. This is just $-9.28.35c_1(N)$, which is nonzero in $H^2(\mathcal{A}_3(l); \mathbb{Q})$. Consequently, the extension is non-trivial as claimed. $\quad \square$

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